$$\varphi(x) = \max_{\tilde{y} \in \Phi(x)} f(x, \tilde{y})$$

is continuous, and the m-mapping $F:X \to K(Y)$

$$F(x) = \{y \mid y \in \Phi(x),\ f(x, y) = \varphi(x)\}$$

is upper semicontinuous.

3. Continuous Sections and Single-Valued Approximations of m-Mappings

Let X, Y be topological spaces, and let $f:X \to Y$ be an m-mapping.

1.3.1. Definition. A continuous, single-valued mapping $f:X \to Y$ is called a continuous section of an m-mapping F if

$$f(x) \in F(x)$$

for all $x \in X$.

The existence of continuous sections is closely connected with lower semicontinuity of a multivalued mapping. The following assertion characterizes this fact.

1.3.2. THEOREM. Let $F:X \to P(Y)$ be an m-mapping. If for any points $x \in X$ and $y \in F(x)$ there exists a continuous section $f:X \to Y$ of the m-mapping F such that $f(x) = y$, then F is a lower semicontinuous m-mapping.

Michael's theorem is one of the basic results of the theory of continuous sections which has found many applications.

1.3.3. THEOREM. The following properties of a T_1-space X are equivalent:

a) X is paracompact;

b) if Y is a Banach space, then each lower semicontinuous m-mapping $F:X \to Cv(Y)$ has a continuous section.

The proof of Theorem 1.3.3 is based on the following assertion.

1.3.4. LEMMA. Let X be a paracompact space, and let Y be a normal space; let $F:X \to Cv(Y)$ be a lower semicontinuous m-mapping; then for any $\varepsilon > 0$ there exists a continuous single-valued mapping $f_\varepsilon:X \to Y$ such that $f_\varepsilon(x) \in U_\varepsilon(F(x))$ for any $x \in X$.

This mapping f_ε is naturally called an ε-section of the m-mapping F.

There are many examples which show that the conditions of completeness of the space Y, closedness and convexity of the range of the m-mapping F, and the condition of lower semicontinuity of this mapping are essential for the existence of a continuous section. However, it is obvious that there exist m-mappings which are not lower semicontinuous but have a continuous section. We shall consider the problem of the existence of a continuous section in terms of the local structure of m-mappings (see [22]).

Let X be a metric space, Y be a convex compact subset of the Banach space E, and let $F:X \to Kv(Y)$ be some m-mapping. We set $F^\varepsilon(x) = U_\varepsilon(F(x)) \cap Y$. For each point $x_0 \in X$ we define the set $L(F)(x_0)$ by the rule

$$L(F)(x_0) = \bigcap_{\varepsilon>0} \left(\overline{\bigcup_{\delta>0} \left(\bigcap_{x \in U_\delta(x_0)} F^\varepsilon(x) \right)} \right).$$

1.3.5. THEOREM. In order that an m-mapping $F:X \to Kv(Y)$ have an ε-section for any $\varepsilon > 0$ it is necessary and sufficient that $L(F)(x_0) \neq \emptyset$ for any $x_0 \in X$.

We remark that nonemptiness of the set $L(G)(x)$ for any $x \in X$ does not yet guarantee the presence of a continuous section of an m-mapping F.

We consider iterations of L:

$$L^0(F) = F,\ L^n(F) = L(L^{n-1}(F)),\ n \geqslant 1.$$

We continue this process for each transfinite number of first type, while for a transfinite number of second type we set

$$L^\alpha(F)(x) = \bigcap_{\beta<\alpha} L^\beta(F)(x).$$

We shall say that the sequence $\{L^{\alpha}(F)\}$ stabilizes at step α_0 if

$$L^{\alpha_0}(F)(x) = L^{\alpha_0+1}(F)(x)$$

for any $x \in X$.

1.3.6. THEOREM. In order that an m-mapping $F:X \to Kv(Y)$ have a continuous section it is necessary and sufficient that the sequence $\{L^{\alpha}(F)\}$ stabilize at some transfinite step α_0 and $L^{\alpha_0}(F)(x) \neq \emptyset$ for any $x \in X$.

If we consider m-mappings with nonconvex ranges, then the problem of the existence of a continuous section becomes much more difficult. It is possible to give an example of a continuous m-mapping with a contractible range not having a continuous section.

1.3.7. Example. Suppose the m-mapping $F:[-1, 1] \to K(\mathbf{R}^2)$ is defined by the rule

$$F(x) = \begin{cases} \Gamma_{\left[\frac{1}{2}x, x\right]}, & \text{if} \quad x \neq 0, \\ \{(0, y) \mid -1 \leqslant y \leqslant 1\}, & \text{if} \quad x = 0, \end{cases}$$

where $\Gamma_{\left[\frac{1}{2}x, x\right]}$ is the graph of the function $y = \sin\frac{1}{x}$ on the interval $\left[\frac{1}{2}x, x\right]$. This mapping has an ε-section for any $\varepsilon > 0$ but it does not have a continuous section.

We shall construct an obstruction to the ε-section property of one class of m-mappings.

Let X be a compact metric space, and let Y be a metric space.

1.3.8. LEMMA. Let $F:X \to P(Y)$ be a lower semicontinuous m-mapping with nonempty compact images. Then for any numbers $\varepsilon > 0$, $\beta > 0$ there is a positive number $\alpha = \alpha(\varepsilon, \beta)$ such that in a β-neighborhood of any set T of diameter less than α there is a point x_0 such that $F(x_0) \subset F_\varepsilon(x) = U_\varepsilon(F(x))$ for any $x \in T$.

We call a point $x_0 \in X$ satisfying the conditions of Lemma 1.3.8 a companion of the set T.

Let X be a finite polyhedron of dimension n, let Y be a compact metrix space, and let $F:X \to K(Y)$ be an m-mapping with nonempty compact images.

1.3.9. Definition. We call an m-mapping $\hat{F}:\hat{X} \to K(Y)$ a steplike ε-approximation of the m-mapping F if there exists a triangulation $\mathscr{K}$ of the polyhedron X such that the following conditions are satisfied.

a) $\hat{F}(x) \subset F^{\varepsilon}(x)$ for any $x \in X$;

b) on any simplex σ^i of the triangulation $\mathscr{K}$ the m-mapping $\hat{F}$ is constant, i.e., $\hat{F}(x) = A_i$ for any $x \in \sigma^i$;

c) if $\sigma^i \subset \partial\sigma^j$, then $\hat{F}(x) \subset \hat{F}(y)$ for $x \in \sigma^i$, $y \in \sigma^j$.

1.3.10. THEOREM. Let $F:X \to K(Y)$ be a lower semicontinuous m-mapping with nonempty compact images. Then for any $\varepsilon > 0$ the m-mapping F has a steplike ε-approximation $\hat{F}$.

Proof. We consider sequences of numbers $\{\varepsilon_i\}_{i=1}^{n+1}$, $\{\beta_i\}_{i=1}^{n}$ and a number d_0 satisfying the following relations:

$$0 < \varepsilon_1 < \varepsilon_2 < \ldots < \varepsilon_n < \varepsilon_{n+1} < \varepsilon,$$

$$0 < \beta_{i+1} < \frac{1}{4}\beta_i, \quad 4\beta_{i+1} + 2d_0 < \alpha(\varepsilon_{i+1} - \varepsilon_i;\ \beta_i),$$

where $i = 1, 2, \ldots, n$. We remark that such sequences can always be constructed for any number $\varepsilon > 0$ in the following manner: the sequence $\{\varepsilon_i\}_{i=1}^{n+1}$ is prescribed arbitrarily, while the construction of $\{\beta_i\}_{i=1}^{n}$ and d_0 is realized by beginning with β_1 and proceeding upward along the inequalities.

We triangulate the polyhedron so finely that the diameter of each simplex is less than $\min(d_0;\ \alpha(\varepsilon_{n+1} - \varepsilon_n;\ \beta_n))$. This triangulation is the desired triangulation $\mathscr{K}$. We construct the m-mapping $\hat{F}$ successively, beginning with simplices of dimension n.

Let σ^n be an n-dimensional simplex of the triangulation $\mathscr{K}$; then $\operatorname{diam}\sigma^n < \alpha(\varepsilon_{n+1} - \varepsilon_n, \beta_n)$, and hence in a β_n-neighborhood of σ^n there is a companion of this symplex – a point x^* – such that $F(x^*) \subset F^{\varepsilon_{n+1}-\varepsilon_n}(x)$ for any $x \in \sigma^n$. Then $F^{\varepsilon_n}(x^*) \subset F^{\varepsilon_{n+1}}(x)$, and hence the set $A = F^{\varepsilon_n}(x^*) \subset F^{\varepsilon_{n+1}}(x)$. We now set $\hat{F}(x) = A$ for any $x \in \sigma^n$. We carry out analogous constructions with all n-dimensional simplices of the triangulation $\mathscr{K}$.

We consider an (n − 1)-dimensional simplex σ^{n-1}. Suppose this simplex is the face of the n-dimensional simplices $\sigma_1^n, \sigma_2^n, \ldots, \sigma_k^n$. Let the points $x_1^*, x_2^*, \ldots, x_k^*$ be the companions of the corresponding n-dimensional simplices. We set $T^{n-1}=\sigma^{n-1}\cup\left(\bigcup_{i=1}^{k} x_i^*\right)$; we shall estimate the diameter of this set:

$$\operatorname{diam} T^{n-1} \leqslant 2d_0+2\beta_n<\alpha(\varepsilon_n-\varepsilon_{n-1}, \beta_{n-1}).$$

Hence there exists a point $x^*\in U_{\beta_{n-1}}(T^{n-1})$ – a companion of this set – such that $F(x^*)\subset F^{\varepsilon_n-\varepsilon_{n-1}}(x)$ for any $x\in T^{n-1}$, i.e.,

$$F^{\varepsilon_{n-1}}(x^*)\subset F^{\varepsilon_n}(x) \quad \text{for any} \quad x\in\sigma^{n-1};$$

$$F^{\varepsilon_{n-1}}(x^*)\subset F^{\varepsilon_n}(x_i^*)=A_i \quad \text{for any} \quad i=1, 2, \ldots, k.$$

We set $F^{\varepsilon_{n-1}}(x^*)=A$, and we set $\hat{F}(x) = A$ for any $x\in\sigma^{n-1}$. We define $\hat{F}$ similarly for the remaining (n − 1)-dimensional simplices.

Suppose the m-mapping $\hat{F}$ has been constructed on simplices of dimension n, n − 1,...,k + 1.

We now consider a k-dimensional simplex σ^k. Suppose this simplex is a factor of the (k + 1)-dimensional simplices $\sigma_1^{k+1}, \sigma_2^{k+1}, \ldots, \sigma_s^{k+1}$ and the points $x_1^*, x_2^*, \ldots, x_s^*$ are the companions of these simplices. We set $T^k=\sigma^k\cup\left(\bigcup_{i=1}^{s} x_i^*\right)$. In this case

$$\operatorname{diam} T^k\leqslant 2d_0+2\beta_n+\ldots+2\beta_{k+1}\leqslant 2d_0+3\beta_{k+1}<\alpha(\varepsilon_{k+1}-\varepsilon_k, \beta_k).$$

Then there is a point $x^*\in U_{\beta_k}(T^k)$, which is a companion of this set, such that $F(x^*)\subset F^{\varepsilon_{k+1}-\varepsilon_k}(x)$ for any $x\in T^k$, i.e.,

$$F^{\varepsilon_k}(x^*)\subset F^{\varepsilon_{k+1}}(x) \quad \text{for any} \quad x\in\sigma^k;$$

$$F^{\varepsilon_k}(x^*)\subset F^{\varepsilon_{k+1}}(x_i^*)=A_i \quad \text{for any} \quad i=1, 2, \ldots, s.$$

We set $F^{\varepsilon_k}(x^*) = A$ and $\hat{F}(x) = A$ for any $x\in\sigma^k$. We define $\hat{F}$ similarly for all the remaining k-dimensional simplices and in the remaining dimensions. The m-mapping constructed $\hat{F}$ satisfies all the conditions of the theorem. The theorem is proved.

An analogous construction for steplike ε-approximations of upper semicontinuous m-mappings was proved in the work [31] and found further development in the works [11, 23].

We note that any section of an m-mapping $\hat{F}$ is an ε-section of the m-mapping F. Hence, in order that F be ε-selective, it suffices to prove selectivity of steplike ε-approximations $\hat{F}$. We shall now construct an obstruction to the existence of continuous actions of the m-mapping $\hat{F}$ (see also [18]).

Suppose some triangulation $\mathscr{K}$ is fixed on the polyhedron X. We consider an m-mapping $\hat{F}: X \to P(Y)$ satisfying the following conditions:

1) on any simplex σ^i of the triangulation $\mathscr{K}$ the m-mapping $\hat{F}$ is constant;
2) if $\sigma^i\subset\partial\sigma^{i+1}$, then $\hat{F}(\sigma^i)\subset\hat{F}(\sigma^{i+1})$, and the inclusion mapping induces an isomorphism of the homotopy groups in dimensions j = 0, 1,...,n − 1;
3) the sets $\hat{F}(x)$ for any $x\in X$ are (n − 1)-simple.

Then the following lemma holds.

<u>1.3.11. LEMMA.</u> If the polyhedron X is linearly connected, then for any two sets $\hat{F}(\sigma_1^{k_1})$ and $\hat{F}(\sigma_2^{k_2})$ there exists an isomorphism $i_{12}: \pi_j(\hat{F}(\sigma_1^{k_1})\to\pi_j(\hat{F}(\sigma_2^{k_2})), 0\leqslant j\leqslant n-1$. If the polyhedron is simply connected, then the isomorphism i_{12} is canonical, i.e., for any set $\hat{F}(\sigma_3^{k_3})$ the following diagram is commutative:

$$\begin{array}{ccc} \pi_j(\hat{F}(\sigma_1^{k_1})) & \xrightarrow{i_{12}} & \pi_j(\hat{F}(\sigma_2^{k_2})) \\ & {\scriptstyle i_{13}}\nwarrow \quad \nearrow{\scriptstyle i_{32}} & \\ & \pi_j(\hat{F}(\sigma_3^{k_3})). & \end{array}$$

We note that in the case $\pi_j(\hat{F}(\sigma^k))=0$ for $0\leqslant j\leqslant n-1$ the trivial isomorphism i_{12} is canonical for any polyhedron X.

Suppose the polyhedron X and the m-mapping $\hat{F}$ are such that the homomorphism $i_{12}\colon \pi_j(\hat{F}(\sigma_1^{k_1}))\to$ is canonical for j = 0, 1,...,n − 1. Suppose a continuous section f of the m-mapping $\hat{F}$ is defined on the (ℓ − 1)-skeleton $\mathscr{K}^{(l-1)}$ of the polyhedron $\mathscr{K}$. We construct an obstruction to the continuation of this section to the ℓ-skeleton $\mathscr{K}^{(l)}$.

Let σ^{ℓ} be an arbitrary ℓ-dimensional simplex. We consider the composition of mappings

$$S^{l-1}\xrightarrow{\varkappa}\partial\sigma^{l}\xrightarrow{f}\hat{F}(\sigma^{l}),$$

where $\varkappa$ is an arbitrary homeomorphism. Then to the simplex σ^{ℓ} it is possible to assign an element $[f\circ\varkappa]$ of the homotopy group $\pi_{l-1}(\hat{F}(\sigma^{l}))$. We fix some set $\hat{F}(\sigma^{s_0})$ and denote it by Y_0. Then by Lemma 1.3.11 there exists a canonial isomorphism $i\colon \pi_{l-1}(\hat{F}(\sigma^{l}))\to\pi_{l-1}(Y_0)$. We assign to the simplex σ^{ℓ} the element $i([f\circ\varkappa])\in\pi_{l-1}(Y_0)$.

A mapping of set of ℓ-dimensional simplices of the polyhedron $\mathscr{K}$ into the group $\pi_{\ell-1}(Y_0)$ has thus been constructed.

By extending this mapping to ℓ-dimensional chains, we obtain a cochain $c_f^l\in C^l(\mathscr{K},\pi_{l-1}(Y_0))$.

<u>1.3.12. Definition.</u> The cochain c_f^{ℓ} is called an ℓ-obstruction to the continuation of the section f.

The obstruction c_f^{ℓ} possesses properties analogous to the properties of a classical obstruction.

<u>1.3.13. THEOREM.</u> The obstruction c_f^{ℓ} is a cocycle. If the cocycle c_f^{ℓ} is homotopic to zero, then there exists a mapping which is a section of the m-mapping $\hat{F}$ onto the (ℓ − 1)-dimensional skeleton $\mathscr{K}^{(l-1)}$ coincides with f on $\mathscr{K}^{(l-2)}$, and can be continued as a section of $\hat{F}$ to $\mathscr{K}^{(l)}$.

<u>1.3.14. COROLLARY.</u> If the polyhedron X is contractible to a point and the m-mapping $\hat{F}$ satisfies conditions 1, 2, and 3, then the m-mapping $\hat{F}$ has a continuous section.

<u>Proof.</u> The section f is constructed inductively over the skeletons of the polyhedron X.

We consider the zero-dimensional skeleton $\mathscr{K}^{(0)}$. Let σ^0 be an arbitrary zero-dimensional simplex; then for the image $f(\sigma^0)$ it is possible to take an arbitrary point in the set $\hat{F}(\sigma^0)$.

We suppose that the mapping f is a section of $\hat{F}$ on the skeleton $\mathscr{K}^{(l-1)}$, $l>1$. We consider the obstruction $c_f^l\in C^l(\mathscr{K},\pi_{l-1}(Y_0))$. Since $H^{\ell}(X, G) = 0$ for any group of coefficients G, it follows that $c_f^l\sim 0$. By Theorem 1.3.13 there then exists a section $g\colon\mathscr{K}^{(l)}\to Y$. Continuing this process, we obtain a section of the m-mapping $\hat{F}$.

<u>1.3.15. COROLLARY.</u> If $\hat{F}$ satisfies conditions 1, 2, 3 and $\pi_j(\hat{F}(\sigma^l))=0$, $j\in\overline{0,\ n-1}$, $l\in\overline{0,\ n}$, then a section of the m-mapping $\hat{F}$ exists on any finite n-dimensional polyhedron X.

The proof of this corollary is analogous to the proof of Corollary 1.3.14.

<u>1.3.16. Definition.</u> A lower semicontinuous m-mapping $F\colon X\to K(Y)$ with compact images is called homotopically continuous is for any point $x_0\in X$ the following conditions are satisfied:

a) there exists $\varepsilon_0 > 0$ such that for any ε, $0\leqslant\varepsilon\leqslant\varepsilon_0$, the set $F^{\varepsilon}(x_0)$ is (n − 1)-simple and the inclusion mapping $F(x_0)\hookrightarrow F_{\varepsilon}(x_0)$ induces an isomorphism of the homotopy groups in dimensions 0, 1,...,n − 1;

b) for any ε, $0<\varepsilon\leqslant\varepsilon_0$ there exists $\delta = \delta(\varepsilon, x_0)$ such that if $\rho(x, x_0) < \delta$ the inclusion mapping $F(x_0)\hookrightarrow F^{\varepsilon}(x)$ induces an isomorphism of the homotopy groups in dimensions 0, 1,...,n − 1.

We note that if $\pi_j(F(x_0)) = 0$, j = 0, 1,...,n − 1, then condition a) implies condition b).

<u>1.3.17. LEMMA.</u> Let $F\colon X\to K(Y)$ be a homotopically continuous m-mapping; then the m-mapping has a steplike ε-approximation $\hat{F}$, $0 < \varepsilon < \varepsilon_0$ satisfying conditions 1, 2, 3.

<u>Proof.</u> We construct a steplike ε-approximation using the construction of Theorem 1.3.10. That conditions 1 and 3 are satisfied follows from the definition of homotopic continuity and the construction of a steplike ε-approximation. We shall prove that condition 2 is satisfied. Let $0 < \varepsilon < \varepsilon_0$; we consider a positive number $\varepsilon' < \varepsilon_1$. We set $V_{\delta}(x)=\{x'\mid x'\in X,\ \rho(x, x')<\delta(\varepsilon', x)\}$, where $\delta(\varepsilon', x)$ is determined from condition b) of homotopic continuity; then the family

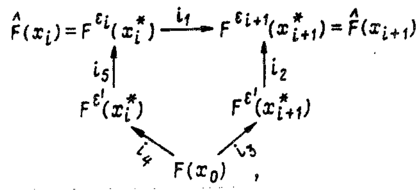
$\{V_\lambda(x)\}_{x\in X}$ forms an open covering of space X. Let r be the Lebesgue number of this covering. On β_1 and d_0 we impose the following additional condition: $4\beta_1 + 2d_0 < r$. We shall now show that if the numbers $\{\varepsilon_i\}_{i=1}^{n+1}$, $\{\beta_i\}_{i=1}^{n}$, d_0 satisfy this additional condition, then $\hat{F}$ satisfies condition 2.

Let $\sigma^i\subset\partial\sigma^{i+1}$; then $\hat{F}(x_i)=F^{\varepsilon_i}(x_i^*)\subset F^{\varepsilon_{i+1}}(x_{i+1}^*)=\hat{F}(x_{i+1})$, where x_i, x_{i+1} are arbitrary points of the simplices σ^i, σ^{i+1}, respectively; x_i^*, x_{i+1}^* are the companions of the corresponding sets. We estimate the distance between x_i^* and x_{i+1}^*:

$$\rho(x_i^*, x_{i+1}^*)\leqslant\rho(x_i^*, T^i)+\operatorname{diam}T^i\leqslant 2d_0+4\beta_i<r.$$

Hence, there exists a point x_0 such that $F(x_0)\subset F^{\varepsilon'}(x_i^*)$ and $F(x_0)\subset F^{\varepsilon'}(x_{i+1}^*)$.

We now consider the following diagram:



where all the mappings are generated by the corresponding imbeddings. Since the mappings i_j, $j = 2, 3, 4, 5$ induce isomorphisms of the homotopy groups in the corresponding dimensions, it follows that i_1 induces an isomorphism in these same dimensions. The lemma is proved.

The next assertions follow from Lemma 1.3.17 and Corollaries 1.3.14 and 1.1.15.

1.3.18. THEOREM. If an n-dimensional polyhedron X is contractible to a point and the m-mapping $F:X \to K(Y)$ is homotopically continuous, then F is ε-selectable.

1.3.19. THEOREM. If the m-mapping $F:X \to K(Y)$ is homotopically continuous and $\pi_j(F(x)) = 0$, $j\in\overline{0, n-1}$, for any point $x\in X$, then F is ε-selectable on any finite n-dimensional polyhedron.

It is easy to see that semicontinuous and closed m-mappings do not admit, generally speaking, continuous sections. Single-valued approximations open the way to the study of their properties.

Let (X, ρ_X), (Y, ρ_Y) be metric spaces. We define a metric ρ in the product of the spaces $X \times Y$ by the equality

$$\rho((x, y), (x', y'))=\max\{\rho_X(x, x');\ \rho_Y(y, y')\}.$$

1.3.20. Definition. Let $F:X \to C(Y)$ be some m-mapping. A multivalued mapping $F_\varepsilon:X \to C(Y)$, where $\varepsilon > 0$, is called a multivalued ε-approximation of the m-mapping F if

$$\rho_*(\Gamma_X(F_\varepsilon), \Gamma_X(F))=\sup_{z\in\Gamma_X(F_\varepsilon)}\rho(z, \Gamma_X(F))<\varepsilon,$$

i.e., the graph $\Gamma_X(F_\varepsilon)$ belongs to an ε-neighborhood of the graph $\Gamma_X(F)$.

If F_ε is a single-valued continuous mapping, then it is said that it is a single-valued ε-approximation of the m-mapping F. The question of the existence of single-valued ε-approximations is important for applications. This can be illustrated by the following example.

Let (X, ρ_X) be a compact metric space, let (Y, ρ_Y) be a metric space, and let $F:X \to C(Y)$ be a closed m-mapping; let y_0 be an arbitrary point of Y.

1.3.21. THEOREM. If for any $\varepsilon > 0$ there exists a single-valued ε-approximation $f_\varepsilon:X \to Y$ of the m-mapping F such that the equation $f_\varepsilon(x) = y$ has a solution, then there exists a point $x_0\in X$, which is a solution of the operator inclusion $y_0\in F(x)$.

The next assertion [70] is one of the basic results on the existence of single-valued ε-approximations.

1.3.22. THEOREM. Let X be a metric space, and let Y be a metric lcs. Then any upper semicontinuous m-mapping $F:X \to Cv(Y)$ for any $\varepsilon > 0$ possesses an ε-approximation $f_\varepsilon:X \to Y$ such that

$$f_\varepsilon(X)\subset\operatorname{co}F(X).$$